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An analytical study of heat and mass transfer through a parallel-plate channel with recycle

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Abstract—The effects of recycle at the ends on the heat and mass transfer through a parallel-plate channel with uniform wall temperature are studied by an orthogonal expansion technique. The heat and mass transfer problem is solved for fully developed laminar velocity profiles in a parallel-plate channel with ignoring axial conduction, and with fluid properties which are temperature independent. Analytical results show that recycle can enhance the heat transfer rate for high Graetz numbers compared with that in an open conduit (without an impermeable plate inserted and without recycle). The preheating and residence-time are the two effects which are in domination of the heat transfer behavior.

1. INTRODUCTION

Heat and mass-transfer processes in a bounded conduit has attracted a great deal of interest in recent years to engineers. The problem of laminar heat and mass transfer at steady state with negligible axial conduction or diffusion is known as the Graetz problem [1, 2]. Indeed, the assumption of negligible axial conduction is not always valid, particularly for low Prandtl number fluids such as liquid metals. Considering a variety of boundary conditions and different geometries is referred to the extended Graetz problem [3–7]. Beyond the initial concern for single-stream problems, the applicability of these results to multi-stream or multiphase problems is expected. Since the equation of heat and/or mass transfer in these problems, so called conjugated Graetz problems, are coupled through mutual conditions at the boundaries [8–15].

Yeh *et al.* [16] has proposed the effects of recycling of fluid at the ends on conjugated Graetz problems, and many separation processes and reactors design have been developed with internal or external refluxes at both ends, such as loop reactors [17, 18], air-lift reactors [19, 20] and draft-tube bubble columns [21, 22], which are widely used in absorption, fermentation, and polymerization. The reflux is a very important factor to be considered in designing those heat and mass transfer processes. The theoretical formulation by use of orthogonal expansion techniques [23–30] was used in this study, the availability of such a simplifying model have been derived will be an important contribution to the design and analysis of heat and mass transfer problems with internal and external refluxes at both ends and mutual conditions at the boundaries.

The purpose of this work is to develop an orthogonal expansion technique to the heat and mass transfer through a parallel-plate channel, and also to show that the method of analysis is applicable to counter-current operation with internal or external recycle at the ends of conduit. The present study includes the effects of recycling on heat and mass transfer and the improvement of transfer efficiency with reflux ratio and Graetz number as parameters.

2. THEORETICAL ANALYSIS

Consider the heat transfer in two channels with thickness ΔW and $(1-\Delta)W$, respectively, which is to divide a parallel conduit with thickness W, length L, and infinite width by inserting an impermeable plate with negligible thickness and thermal resistance as in Fig. 1. Before entering the lower channel, the fluid with volume flow rate V and temperature T_0 will mix the volume flow rate of recycle RV which is controlled by means of a conventional pump situated at the end of upper channel.

After the following assumptions are made: constant physical properties and wall temperatures, purely fully-developed laminar flow in each channel, and negligible axial diffusion as well as entrance length and end effects, the velocity distributions and equations of energy in dimensionless form may be obtained as

$$\frac{\partial^2 \psi_a(\eta_a,\xi)}{\partial \eta_a^2} = \left(\frac{W_a^2 v_a}{L\alpha}\right)\frac{\partial \psi_a(\eta_a,\xi)}{\partial \xi} \qquad (1)$$

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NOMENCLATURE

B	conduit width [m]
c_p	specific heat at constant pressure [kJ kg^{-1} K^{-1}]
d_{mn}	coefficient in the eigenfunction $F_{a,m}$
e_{mn}	coefficient in the eigenfunction $F_{b,m}$
f	constant defined in the velocity distribution
F_m	eigenfunction associated with eigenvalue λ_m
G_z	Graetz number, $VW/\alpha BL$
G	function defined during the use of orthogonal expansion method
$\bar{h}$	average heat transfer coefficient [kW m^{-2} K^{-1}]
I	improvement of the degree of heat transfer, equation (36)
k	thermal conductivity of the fluid [kW m^{-1} K^{-1}]
L	conduit length [m]
$\overline{Nu}$	Nusselt number
S_m	expansion coefficient associated with eigenvalue λ_m
R	reflux ratio, reverse volume flow rate divided by input volume flow rate
T	temperature of fluid [K]
T_0	inlet temperature of fluid in conduit [K]
T_s	wall temperature [K]
V	input volume flow rate of conduit [m^3 s^{-1}]
v	velocity distribution of fluid [m s^{-1}]
W	thickness of conduit [m]
x	longitudinal coordinate [m]
z	transversal coordinate [m].

Greek symbols

α	thermal diffusivity of fluid [m^2 s^{-1}]
Δ	ratio of thickness between forward flow channel and conduit
η	transversal coordinate, x/W
θ	dimensionless temperature
λ_m	eigenvalue
ξ	longitudinal coordinate, z/L
ρ	density of the fluid [kg m^{-3}]
ψ	dimensionless temperature.

Subscripts

a	forward flow channel
b	backward flow channel
L	outlet temperature.

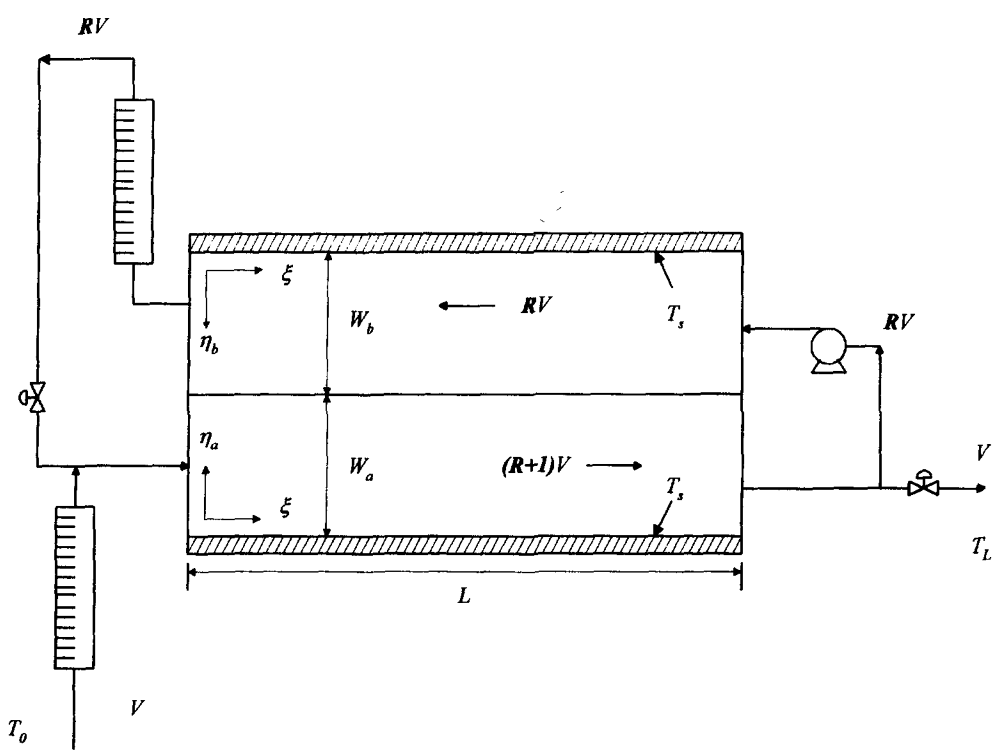

Fig. 1. Parallel conduit with refluxes at both ends.

$$\frac{\partial^2\psi_b(\eta_b,\xi)}{\partial\eta_b^2} = \left(\frac{W_b^2 v_b}{L\alpha}\right)\frac{\partial\psi_b(\eta_b,\xi)}{\partial\xi} \tag{2}$$

$$v_a(\eta_a) = f_a(6\eta_a - 6\eta_a^2), \quad 0 \leqslant \eta_a \leqslant 1 \tag{3}$$

$$v_b(\eta_b) = f_b(6\eta_b - 6\eta_b^2), \quad 0 \leqslant \eta_b \leqslant 1 \tag{4}$$

in which

$$f_a = [(R+1)V/W_aB] = \bar{f}_a(R+1),$$

$$f_b = -[VR/W_bB] = \bar{f}_bR, \quad \eta_a = \frac{x_a}{W_a}$$

$$\eta_b = \frac{x_b}{W_b}, \quad \xi = \frac{z}{L}, \quad \psi_a = \frac{T_a - T_s}{T_0 - T_s},$$

$$\psi_b = \frac{T_b - T_s}{T_0 - T_s}, \quad G_z = \frac{V(W_a + W_b)}{\alpha BL} = \frac{VW}{\alpha BL}$$

$$W_a = \Delta W, \quad W_b = (1-\Delta)W,$$

$$\frac{W_a}{W_b} = \frac{\Delta}{1-\Delta}, \quad \theta_a = 1 - \psi_a = \frac{T_a - T_0}{T_s - T_0}$$

$$\theta_b = 1 - \psi_b = \frac{T_b - T_0}{T_s - T_0}. \tag{5}$$

The boundary conditions for solving equation (1) and (2) are

$$\psi_a(0,\xi) = 0 \tag{6}$$

$$\psi_b(0,\xi) = 0 \tag{7}$$

$$\psi_a(1,\xi) = \psi_b(1,\xi) \tag{8}$$

$$-\frac{\partial\psi_a(1,\xi)}{\partial\eta_a} = \frac{W_a}{W_b}\frac{\partial\psi_b(1,\xi)}{\partial\eta_b} \tag{9}$$

$$\psi_a(\eta_a,1) = \psi_b(\eta_b,1) = \psi_L = \frac{T_L - T_s}{T_0 - T_s}. \tag{10}$$

Inspection of the above equations shows that the inlet conditions for both channels are not specified *a priori* and reverse flow occurs. The analytical solution to this type of problem will be obtained by use of an orthogonal expansion technique.

Separation of variables in the form

$$\psi_a(\eta_a,\xi) = \sum_{m=0}^{\infty} S_{a,m}F_{a,m}(\eta_a)G_m(\xi) \tag{11}$$

$$\psi_b(\eta_b,\xi) = \sum_{m=0}^{\infty} S_{b,m}F_{b,m}(\eta_b)G_m(\xi) \tag{12}$$

applied to equation (1), (2), (6)–(9) leads to

$$G_m(\xi) = e^{-\lambda_m(1-\xi)} \tag{13}$$

$$F''_{a,m}(\eta_a) - \left[\frac{\lambda_m W_a^2 v_a(\eta_a)}{L\alpha}\right]F_{a,m}(\eta_a) = 0 \tag{14}$$

$$F''_{b,m}(\eta_b) - \left[\frac{\lambda_m W_b^2 v_b(\eta_b)}{L\alpha}\right]F_{b,m}(\eta_b) = 0 \tag{15}$$

$$F_{a,m}(0) = 0 \tag{16}$$

$$F_{b,m}(0) = 0 \tag{17}$$

$$S_{a,m}F_{a,m}(1) = S_{b,m}F_{b,m}(1) \tag{18}$$

$$-S_{a,m}F'_{a,m}(\eta_a)_{|\eta_a=1} = \frac{W_a}{W_b}S_{b,m}F'_{b,m}(\eta_b)_{|\eta_b=1} \tag{19}$$

where the primes on $F_{a,m}(\eta_a)$ and $F_{b,m}(\eta_b)$ denote the differentiations with respect to η_a and η_b, respectively.

Without loss of generality, we may assume the eigenfunctions $F_{a,m}(\eta_a)$ and $F_{b,m}(\eta_b)$ to be polynomials and have the forms as follows:

$$F_{a,m}(\eta_a) = \sum_{n=0}^{\infty} d_{mn}\eta_a^n, \quad d_{m0} = 0, \quad d_{m1} = 1 \tag{20}$$

$$F_{b,m}(\eta_b) = \sum_{n=0}^{\infty} e_{mn}\eta_b^n, \quad e_{m0} = 0, \quad e_{m1} = 1. \tag{21}$$

Combination of equations (18) and (19) results in

$$\frac{F_{b,m}(1)}{F_{a,m}(1)} = -\frac{W_a}{W_b}\frac{F'_{b,m}(\eta_b)_{|\eta_b=1}}{F'_{a,m}(\eta_a)_{|\eta_a=1}}. \tag{22}$$

Substituting equations (20) and (21) into equations (14) and (15), all the coefficients $d_{m,n}$ and $e_{m,n}$ may be in terms of eigenvalue λ_m after using equations (16) and (17), as referred to in the Appendix. Therefore, it is easy to solve all eigenvalues from equation (22) and the eigenfunctions associated with the corresponding eigenvalues are also well defined by equations (20) and (21). These eigenvalues include a positive set and a negative set, each of which is required for the counterflow system.

It is easy to find the orthogonality conditions as

$$W_b\int_0^1\left[\frac{W_a^2 v_a(\eta_a)}{L\alpha}\right]S_{a,m}S_{a,n}F_{a,m}F_{a,n}\,\mathrm{d}\eta_a + W_a\int_0^1\left[\frac{W_b^2 v_b(\eta_b)}{L\alpha}\right]S_{b,m}S_{b,n}F_{b,m}F_{b,n}\,\mathrm{d}\eta_b = 0 \tag{23}$$

when $n \neq m$. Since

$$\psi_L = \sum_{m=0}^{\infty} S_{a,m}F_{a,m}(\eta_a) = \sum_{m=0}^{\infty} S_{b,m}F_{b,m}(\eta_b). \tag{24}$$

From the orthogonality conditions, the general expressions for the expansion coefficients may be obtained. Accordingly, we have

$$W_b\int_0^1 \psi_L\left[\frac{W_a^2 v_a(\eta_a)}{L\alpha}\right]S_{a,m}F_{a,m}(\eta_a)\,\mathrm{d}\eta_a + W_a\int_0^1 \psi_L\left[\frac{W_b^2 v_b(\eta_b)}{L\alpha}\right]S_{b,m}F_{b,m}(\eta_b)\,\mathrm{d}\eta_b$$

$$= W_b\int_0^1 S_{a,m}^2\left[\frac{W_a^2 v_a(\eta_a)}{L\alpha}\right]F_{a,m}^2(\eta_a)\,\mathrm{d}\eta_a$$

$$+W_a \int_0^1 S_{b,m}^2 \left[\frac{W_b^2 v_b(\eta_b)}{L\alpha}\right] F_{b,m}^2(\eta_b)\,\mathrm{d}\eta_b. \qquad (25)$$

Equation (25) can be rewritten as

$$\psi_L \left[\frac{W_b S_{a,m}}{\lambda_m}\{F'_{a,m}(1) - F'_{a,m}(0)\} + \frac{W_a S_{b,m}}{\lambda_m}\{F'_{b,m}(1) - F'_{b,m}(0)\}\right]$$
$$= W_b S_{a,m}^2 \left[F_{a,m}(1)\frac{\partial F'_{a,m}(1)}{\partial \lambda_m} - F'_{a,m}(1)\frac{\partial F_{a,m}(1)}{\partial \lambda_m} + F'_{a,m}(0)\frac{\partial F_{a,m}(0)}{\partial \lambda_m}\right]$$
$$+ W_a S_{b,m}^2 \left[F_{b,m}(1)\frac{\partial F'_{b,m}(1)}{\partial \lambda_m} - F'_{b,m}(1)\frac{\partial F_{b,m}(1)}{\partial \lambda_m} + F'_{b,m}(0)\frac{\partial F_{b,m}(0)}{\partial \lambda_m}\right]. \qquad (26)$$

Also,

$$\psi_L = \frac{\int_0^1 v_a W_a B \psi_a(\eta_a, 1)\,\mathrm{d}\eta_a}{V(R+1)} = \frac{1}{(R+1)G_z \Delta}\sum_{m=0}^{\infty}\frac{S_{a,m}}{\lambda_m}\{F'_{a,m}(1) - F'_{a,m}(0)\}$$
$$= \frac{-\int_0^1 v_b W_b B \psi_b(\eta_b, 1)\,\mathrm{d}\eta_b}{VR} = \frac{1}{RG_z(1-\Delta)}\sum_{m=0}^{\infty}\frac{S_{b,m}}{\lambda_m}\{F'_{b,m}(1) - F'_{b,m}(0)\}. \qquad (27)$$

Accordingly, once all the eigenvalues are found the possible associated expansion coefficients, $S_{a,m}$ and $S_{b,m}$ can be calculated from equations (26) and (27) with ψ_L assumed. Now, only ψ_L remain undetermined and it can be examined by equation (29) which is readily obtained from the following overall energy balance on both parallel plates

$$V(1-\psi_L) = \int_0^1 \frac{\alpha BL}{W_a}\frac{\partial \psi_a(0,\xi)}{\partial \eta_a}\mathrm{d}\xi + \int_0^1 \frac{\alpha BL}{W_b}\frac{\partial \psi_b(0,\xi)}{\partial \eta_b}\mathrm{d}\xi$$
$$= \left[\frac{\alpha BL}{W_a}\sum_{m=0}^{\infty} S_{am}F'_{am}(0)\int_0^1 e^{-\lambda_m(1-\xi)}\,\mathrm{d}\xi + \frac{\alpha BL}{W_b}\sum_{m=0}^{\infty} S_{bm}F'_{bm}(0)\int_0^1 e^{-\lambda_m(1-\xi)}\,\mathrm{d}\xi\right] \qquad (28)$$

or

$$\psi_L = 1 - \frac{1}{G_z}\left[\sum_{m=0}^{\infty}\frac{(1-e^{-\lambda_m})}{\lambda_m \Delta}S_{am}F'_{am}(0) + \sum_{m=0}^{\infty}\frac{(1-e^{-\lambda_m})}{\lambda_m(1-\Delta)}S_{bm}F'_{bm}(0)\right]. \qquad (29)$$

After the coefficients, $S_{a,m}$ and $S_{b,m}$ are obtained, the mixed inlet temperature is calculated as follows:

$$\psi_a(\eta_a, 0) = \frac{-\int_0^1 v_b W_b B \psi_b(\eta_b, 0)\,\mathrm{d}\eta_b + V}{V(R+1)}$$
$$= \frac{1}{R+1}\left[1 - \frac{W_b B}{V}\sum_{m=0}^{\infty} S_{b,m} \times e^{-\lambda_m}\int_0^1 v_b(\eta_b)F_{b,m}(\eta_b)\,\mathrm{d}\eta_b\right]$$
$$= \frac{1}{R+1}\left[1 - \frac{W_b B}{V}\left(\frac{L\alpha}{W_b^2}\right) \times \sum_{m=0}^{\infty} S_{b,m}\frac{e^{-\lambda_m}}{\lambda_m}\{F'_{b,m}(1) - F'_{b,m}(0)\}\right]$$
$$= \frac{1}{R+1}\left[1 - \frac{1}{G_z(1-\Delta)} \times \sum_{m=0}^{\infty}\left(\frac{e^{-\lambda_m}S_{b,m}}{\lambda_m}\right)\{F'_{b,m}(1) - F'_{b,m}(0)\}\right]. \qquad (30)$$

3. THE IMPROVEMENT OF TRANSFER EFFICIENCY

The Nusselt number is defined as

$$\overline{Nu} = \frac{\bar{h}W}{k} \qquad (31)$$

where the average heat transfer coefficient is defined as

$$q = \bar{h}(2BL)(T_s - T_0). \qquad (32)$$

Since

$$\bar{h}(2BL)(T_s - T_0) = V\rho c_p(T_L - T_0) \qquad (33)$$

or

$$\bar{h} = \frac{V\rho c_p}{2BL}\left(\frac{T_L - T_0}{T_s - T_0}\right) = \frac{V\rho c_p}{2BL}(1-\psi_L). \qquad (34)$$

Thus

$$\overline{Nu} = \frac{\bar{h}W}{k} = \frac{VW}{2\alpha BL}(1-\psi_L) = 0.5G_z(1-\psi_L). \qquad (35)$$

The improvement of heat exchange by operating with recycle is best illustrated by calculating the percentage increase in heat transfer based on the heat exchanger without impermeable plate and recycle as

Table 1. Eigenvalues and expansion coefficients for $\Delta = 0.5$ and $R = 1$

Gz	m	λ_m	$S_{a,m}$	$S_{b,m}$
1	0	−4.63055	0.018942	0.003783
	1	−9.77039	-7.7×10^{-7}	-2.6×10^{-7}
10	0	−0.46306	1.615395	0.322607
	1	−0.97704	-6.5×10^{-5}	-2.2×10^{-5}
100	0	−0.04631	2.991764	0.597479
	1	−0.09770	-1.2×10^{-4}	-4.1×10^{-5}
1000	0	−0.00463	3.208441	0.640751
	1	−0.00977	-1.3×10^{-4}	-4.4×10^{-5}

$$I = \frac{\overline{Nu}-\overline{Nu}_0(\Delta = 1)}{\overline{Nu}_0(\Delta = 1)}\,\%. \tag{36}$$

The results are shown in Table 2.

4. DISCUSSION AND CONCLUSIONS

The equation of heat transfer through a parallel-plate channel with external recycle has been derived by using the orthogonal expansion technique. The eigenfunctions are expanded in terms of an extended power series. Two eigenvalues and associated expansion coefficients were calculated. Some results for $\Delta = 0.5$ and $R = 1$ are shown in Table 1 with Graetz number as a parameter. It was found that only the first negative eigenvalue is necessary to be considered during the calculation procedure.

Figures 2 and 3 show, respectively, the theoretical and experimental dimensionless inlet temperature $\theta_a(\eta_a, 0)$ of the fluid after mixing the outlet temperature $\theta_a(\eta_a, 1)$, with the reflux ratio as a parameter for $\Delta = 0.5$. The experimental data were taken from Yeh *et al.* [16]. A qualitative agreement is achieved between the analytical solutions and experimental data.

The mixed dimensionless inlet temperature increases with the amount of the reflux fluid (or reflux ratio), and the temperature of reflux fluid increases with the residence time which is inversely proportional to the inlet volume rate (or the Graetz number). Accordingly, it is shown in Fig. 2 that the dimensionless inlet temperature of fluid after mixing increases with reflux ratio but decreases with Graetz number. It is shown in Fig. 3 that for a fixed reflux ratio, the dimensionless average outlet temperature decreases also with increasing the Graetz number owing to the short residence time of fluid.

The methods for improving the performance in heat (or mass) transfer devices are either the increase of residence time or the production of preheating (or premixing) effect. Actually, the application of recycle to heat (or mass) transfer devices creates two conflict effects: the desirable preheating effect of the inlet fluid and the undesirable effect of decreasing residence time. At low Graetz number (either small input volume flow rate V or large conduit length L) the residence time is essentially long and should be kept for good performance. In this case, therefore, the preheating effect by increasing the reflux ratio cannot compensate for the decrease of residence time, and hence the outlet temperature (or heat transfer) decreases with increasing reflux ratio, as shown in Figs. 3 and 4. However, the introduction of reflux still has positive effects on the heat transfer for large Graetz number. This is due to the preheating effect having more influence than the residence-time effect here. Figure 4 shows the theoretical average Nusselt numbers with the reflux ratio as a parameter for $\Delta = 0.5$ while Fig. 5 with reflux ratio and the ratio of the thickness Δ as parameters. It is concluded that recycle can enhance heat and mass transfer for the fluid with large Graetz number.

Since the position of the barrier has much influence on the heat transfer behavior, average Nusselt number has been calculated and presented in Fig. 6 with the ratio of the thickness Δ as a parameter for $Gz = 100$ while the percentage of the improvement of heat transfer were shown in Table 2 with Graetz number, reflux ratio and the ratio of the thickness Δ as parameters. It is shown that the improvement of heat transfer increases with Graetz number and reflux ratio while the average Nusselt number increases as the ratio of the thickness Δ goes away from 0.5, especially for $\Delta < 0.5$.

It is apparent that the mathematical formulation derived in this study may also be applied to other conjugated Graetz problems with external recycle at both ends.

Table 2. The improvement of the transfer efficiency with reflux ratio and barrier position as parameters

	$R = 1.0$			$R = 2.0$			$R = 5.0$		
I (%)	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 0.75$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 0.75$	$\Delta = 0.25$	$\Delta = 0.50$	$\Delta = 0.75$
$G_z = 1$	−0.01	−0.21	−0.20	−0.08	−0.59	−0.53	−0.48	−1.54	−1.29
$G_z = 10$	22.15	6.06	13.70	22.83	6.93	13.94	23.10	7.58	14.17
$G_z = 100$	124.61	49.85	89.31	149.72	62.55	100.60	173.26	75.49	112.73
$G_z = 1000$	162.50	89.31	113.78	205.09	78.34	131.25	248.83	97.03	150.78

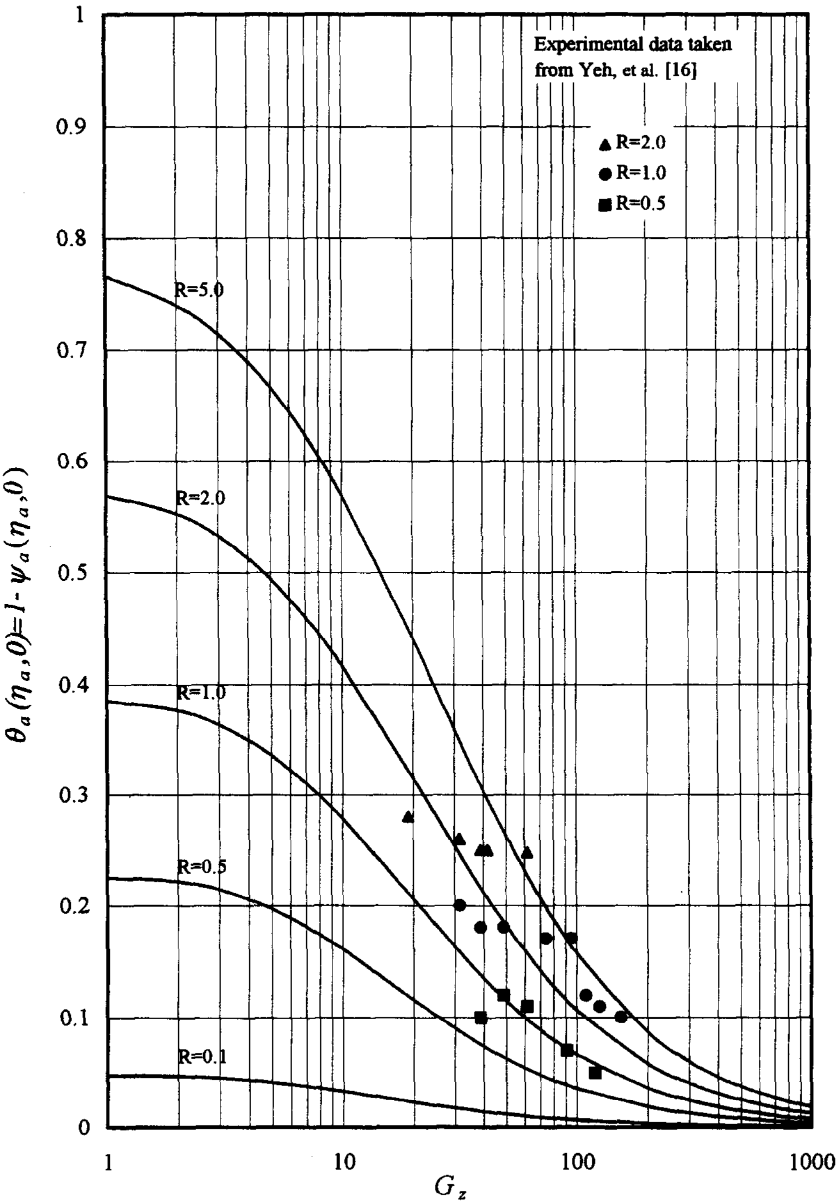

Fig. 2. Theoretical and experimental dimensionless inlet temperature of fluid after mixing. Reflux ratio as a parameter; $\Delta = 0.5$, $0 \leqslant \eta_a \leqslant 1$.

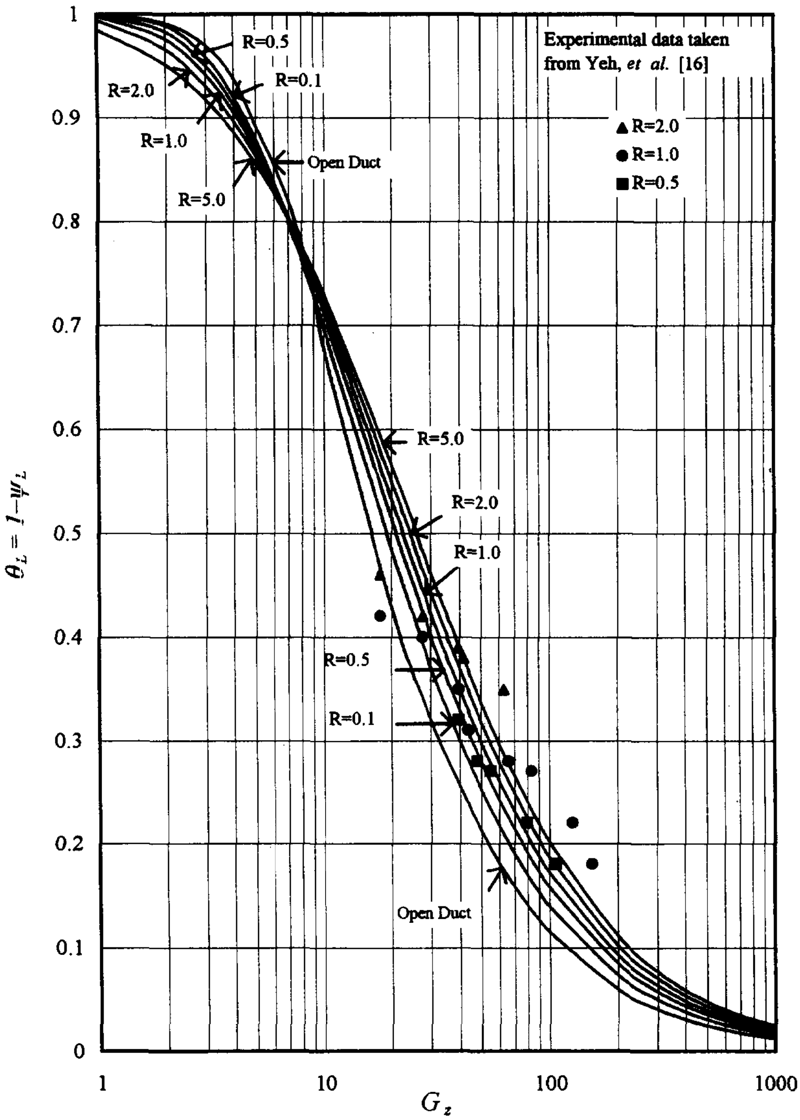

Fig. 3. Theoretical and experimental dimensionless average outlet temperature. Reflux ratio as a parameter; $\Delta = 0.5$.

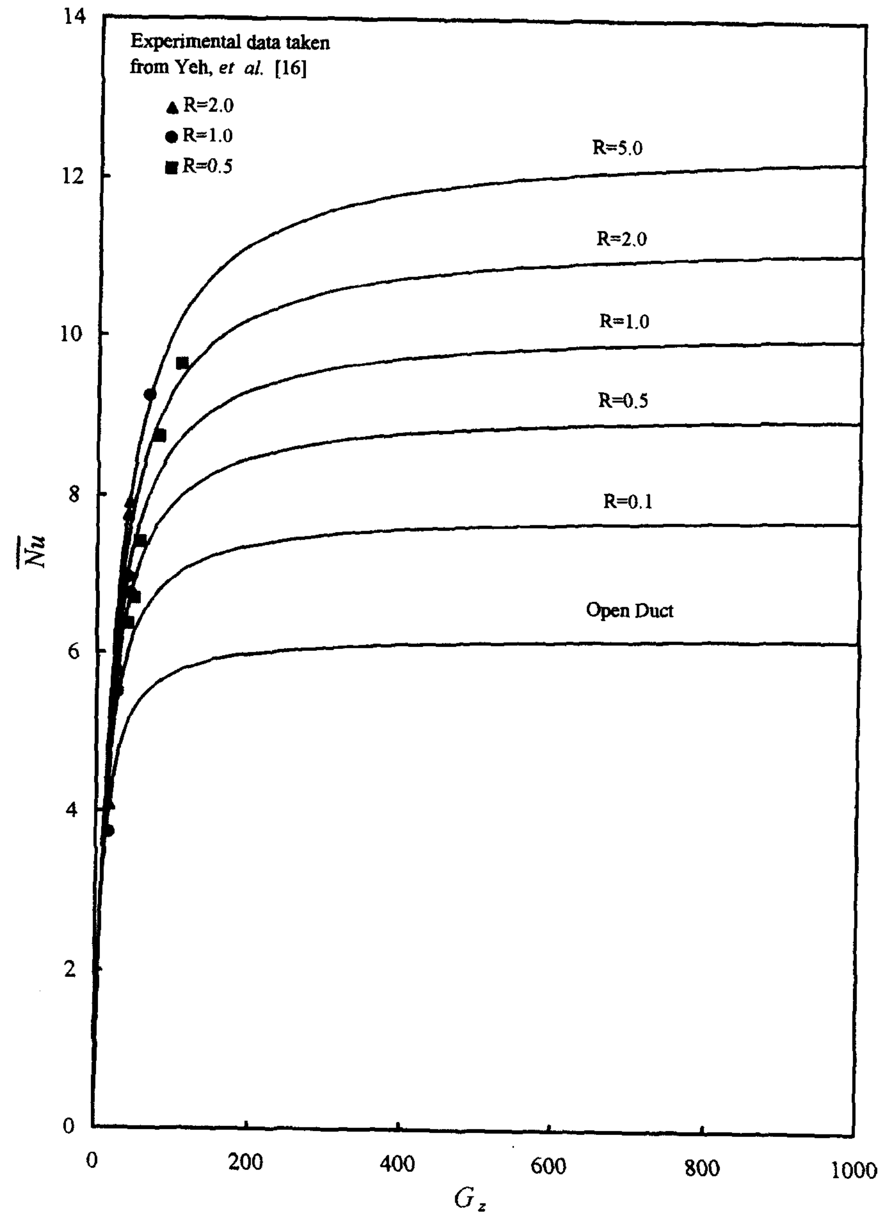

Fig. 4. Theoretical and experimental average Nusselt number. Reflux ratio as a parameter; $\Delta = 0.5$.

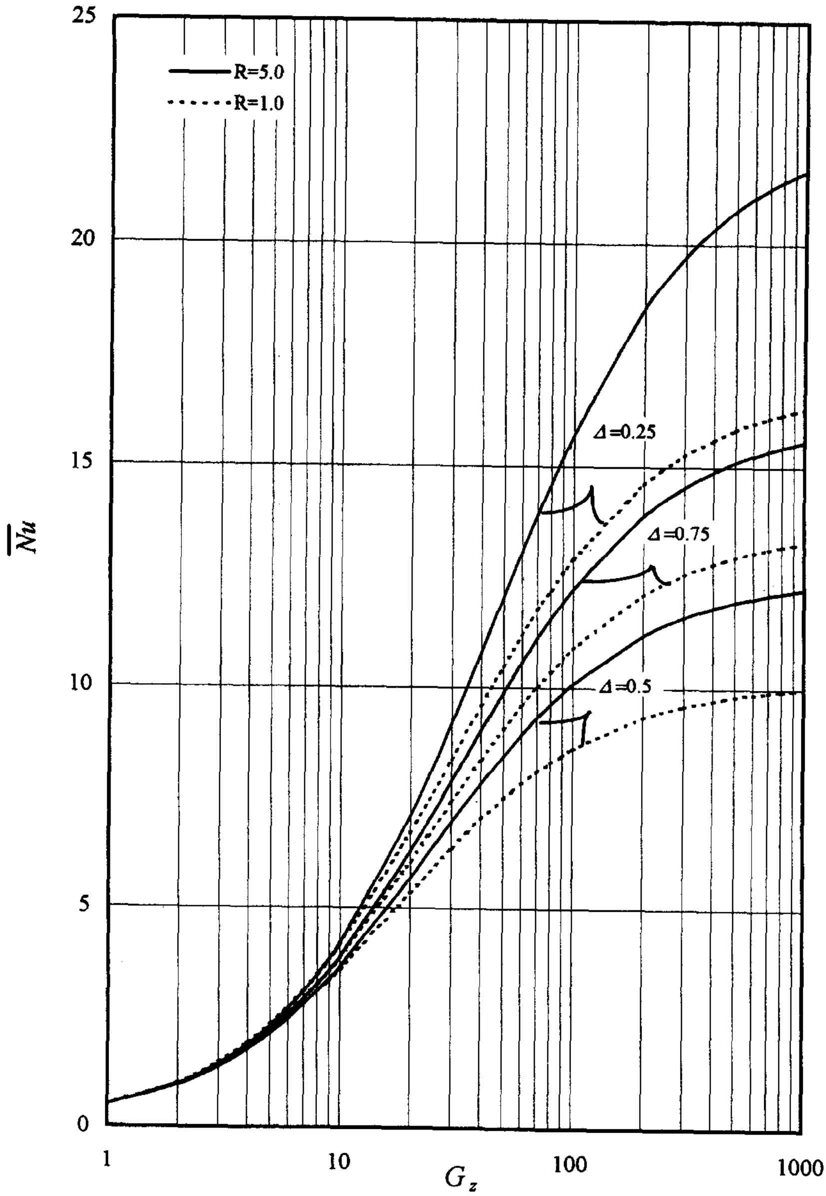

Fig. 5. Average Nusselt number. Reflux ratio and Δ as parameters.

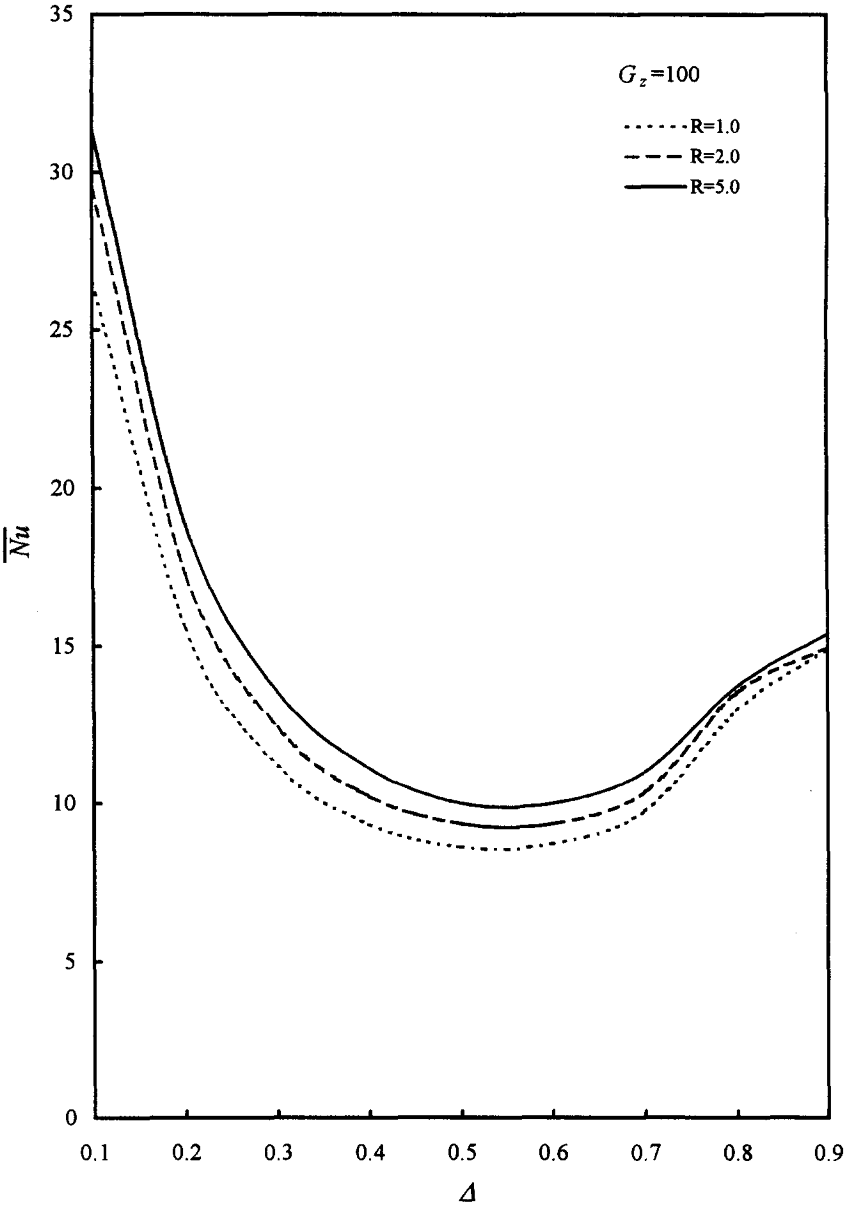

Fig. 6. Average Nusselt number at various barrier positions with reflux ratio as a parameter; $G_z = 100$.

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APPENDIX

Equations (14) and (15) can be rewritten as

$$F''_{a,m}(\eta_a)-\lambda_m G_z(R+1)\,\Delta(6\eta_a-6\eta_a^2)F_{a,m}(\eta_a)=0 \quad \text{(A1)}$$

$$F''_{b,m}(\eta_b)+\lambda_m G_z R(1-\Delta)(6\eta_b-6\eta_b^2)F_{b,m}(\eta_b)=0. \quad \text{(A2)}$$

Combining equations (A1), (A2), (16), (17), (20) and (21) yields

$$\begin{aligned}
d_{m1} &= 1\\
d_{m2} &= 0\\
d_{m3} &= 0\\
d_{m4} &= \tfrac{1}{2}\lambda_m G_z(R+1)\,\Delta\\
&\vdots\\
d_{mn} &= \frac{6}{n(n-1)}\lambda_m G_z(R+1)\,\Delta(d_{m(n-3)}-d_{m(n-4)}) \quad \text{(A3)}
\end{aligned}$$

$$\begin{aligned}
e_{m0} &= 0\\
e_{m1} &= 1\\
e_{m2} &= 0\\
e_{m3} &= 0\\
e_{m4} &= -\tfrac{1}{2}\lambda_m G_z R(1-\Delta)\\
&\vdots\\
e_{mn} &= -\frac{6}{n(n-1)}\lambda_m G_z R(1-\Delta)(e_{m(n-3)}-e_{m(n-4)}). \quad \text{(A4)}
\end{aligned}$$